An electric arc in a transverse magnetic field: a theory for low power gradient

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A uniform electric arc column is held at rest against an imposed low-speed flow perpendicular to its length by an applied magnetic field transverse to both the arc and the flow. The situation is represented mathematically by two regions separated by an isothermal boundary, the arc periphery, across which certain gas properties change discontinuously. It is assumed that the arc has low power gradient so that the Nusselt number is small compared with unity. The Reynolds number is then small also and the methods of the theory of flow at low Reynolds number are used to obtain solutions for the temperature, magnetic field, velocity and pressure inside and outside the arc. It is found that the periphery of the arc is a circle and its radius is determined by heat transfer. The flow near the periphery, and the drag of the arc, are found to depend on a final boundary condition at the periphery, the form of which is not yet clear. Several examples of possible flow patterns are given, and it is shown that the arc may be likened to a slippery porous body for which the slipperiness and porousness are governed by the final boundary condition. The electric and magnetic characteristics of the arc are derived and shown to be amenable to examination by experiment and to empirical extension for arcs of higher power gradient.

1. Introduction

The problem of an electric arc in a cross-flow and a transverse magnetic field, surveyed recently by Myers & Roman (1966), has been of physical and engineering interest for many years, and it has for long been apparent that it may be treated mathematically through the equations of continuum magneto-fluid-dynamics with the inclusion of heat transfer effects. However, no solutions giving detailed distributions of temperature, velocity and pressure inside and outside the arc have been reported. This situation is in contrast to the case of arcs in axial flow in constrictor tubes, for which increasingly detailed solutions have been obtained since the analysis of Stine & Watson (1962). The main role of continuum theory in the cross-flow problem has hitherto been to provide similarity parameters to aid the analysis of experimental results for straight columns (Lord 1964; Yas'ko 1964; Dautov & Zhukov 1965) and to yield the unconventional explanation of the retrograde motion of curved columns given by Schrade

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(1965). In this paper an idealized form of the problem is considered and a detailed solution is given. The solution is thought to be sufficiently complete to serve as a prototype for subsequent investigations.

The problem is illustrated in figure 1. A uniform electric arc column is held at rest against an imposed subsonic flow perpendicular to the column by an applied magnetic field which is transverse to both the undisturbed flow and the arc. The total current through the arc I is in the z-direction and is maintained by a uniform

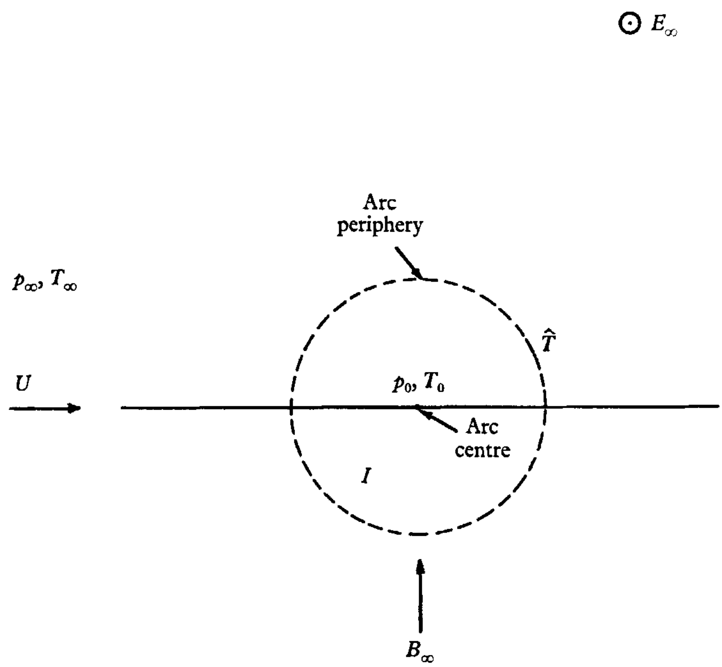


FIGURE 1. Main parameters of problem.

applied electric field (or voltage gradient) E_∞. The undisturbed flow is in the x-direction and has a velocity U which is small compared with the speed of sound appropriate to the ambient pressure p_∞ and the ambient temperature T_∞. The uniform magnetic field B_∞ is applied in the y-direction and this has the effect of holding the arc at rest. To be precise, we view the arc from a frame of reference at rest with respect to the point of maximum temperature within the arc so that this point, the arc centre, stays fixed for all values of U and I, and for all values of p_∞ and T_∞. However, the effective size of the arc does depend on U and I and on p_∞ and T_∞. So, too, do the values of E_∞ and B_∞; we refer to the result for E_∞ in terms of I, U, p_∞ and T_∞ as the electric characteristic of the arc and to the result for B_∞ as the magnetic characteristic. It is implicit in this description of the problem that, for a given gas, the electric and magnetic characteristics are uniquely determined by the current, the flow velocity and the ambient pressure and temperature.

A mathematical model is constructed by first assuming the existence of a temperature $\hat{T}$, such that the electric conductivity is identically zero for all temperatures below $\hat{T}$ and is non-zero for all temperatures above $\hat{T}$; this temperature $\hat{T}$, regarded as an empirical constant for a given gas, serves to define the periphery of the arc as a given isothermal and thus separates the inside of the arc from the outside. Secondly, it is assumed that the electric conductivity, density, viscosity and ratio of specific heat to thermal conductivity take constant values outside the arc which are related to the ambient and peripheral conditions, and constant values inside the arc related to conditions at the periphery and the arc centre. This mathematical model is clearly artificial, but successful theories for arcs in axial flow have been based on the concept of the periphery and the other simplifications are defended on empirical grounds. Indeed, the theory is specifically designed to be used in conjunction with experimental measurements.

The introduction of the arc periphery and the assumption of constant properties outside and inside the arc are not sufficient to define a problem which can be solved in its entirety for arbitrarily large values of current. In this paper we make the extra assumption that the current is sufficiently low for the power input per unit length to the arc (the power gradient) $E_\infty I$ to be small compared with a representative difference of heat-flux potential between the arc and the surrounding flow. In the absence of radiation from the arc, which may be taken for granted for arcs of low power gradient, $E_\infty I$ is equal to the amount of heat being conducted across the arc periphery (since convection through a closed isothermal boundary does not affect the overall energy balance if there are no sources or sinks) and hence the Nusselt number is small compared with one. This is then taken to imply that the size of the arc is small in the sense that the Reynolds number is small compared with one, and hence it is possible to develop a theory of the type familiar for flow at low Reynolds number (Illingworth 1963; Lagerstrom 1964). Here we obtain Oseen solutions outside the arc and Stokes solutions inside the arc and then seek to join the external and internal solutions by satisfying appropriate boundary conditions at the arc periphery.

The joining of the solutions for the temperature is straightforward but the joining of the solutions for the flow is not. The difficulty lies in specifying enough appropriate boundary conditions at the arc periphery to make the flow unique, and the form of the final condition is open to question. The attitude taken in this paper is that, on the assumption that it is correct to expect a unique flow pattern for a given set of values of current, flow velocity and ambient pressure and temperature, it is reasonable that the final boundary condition should not be identifiable at this preliminary stage in the study of the problem. In view of the artificial nature of the periphery and its definition as an isothermal of empirically derived temperature there seems no reason why the final flow condition should take a simple form. Accordingly, rather than apply a speculative condition, the final boundary condition is left open in the analysis and several examples of possible flow patterns are given and their consequences described. However, it is suggested that the form of the final condition should come from an experimental study, and in particular it is shown how a comparison of the theoretically derived arc characteristics with experimentally measured characteristics could shed light on the missing condition.

2. Formulation of mathematical problem

2.1. *Equations*

The following quantities are taken to be uniform outside and inside the arc and discontinuous at the arc periphery: the electric conductivity σ, the density ρ, the viscosity η, and the ratio (c_p/k) where c_p is the specific heat at constant pressure and k is the thermal conductivity. We assume that k is a function of temperature T only and define the heat-flux potential ϕ by $k = d\phi/dT$ with $\phi = 0$ when $T = 0$; the ambient and peripheral values of ϕ, corresponding to T_∞ and $\hat{T}$, are denoted by ϕ_∞ and $\hat{\phi}$. We treat ϕ as the operative temperature variable. The assumption of uniform (c_p/k) means that the enthalpy is linear in ϕ. The equations of steady magneto-fluid-dynamics for a fluid of constant properties (Shercliff 1965), may then be written (in the SI system of units) as:

continuity equation: $$\nabla . \mathbf{v} = 0; \tag{1}$$

momentum equation: $$\rho \mathbf{v} . \nabla \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{j} \times \mathbf{B}; \tag{2}$$

energy equation: $$\rho \mathbf{v} . \nabla[(c_p/k)\phi + \tfrac{1}{2}v^2] = \mathbf{E} . \mathbf{j} + \nabla^2 \phi; \tag{3}$$

Maxwell's equations: $$\nabla \times \mathbf{B} = \mu \mathbf{j}, \tag{4}$$

$$\nabla . \mathbf{B} = 0; \tag{5}$$

Ohm's law: $$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{6}$$

where $\mathbf{v}$ is the velocity, p the pressure, $\mathbf{j}$ the current density, $\mathbf{B}$ the magnetic field, $\mathbf{E}$ the electric field and μ the magnetic permeability of free space (a dimensional constant equal to $4\pi 10^{-7}$ ohm second/metre). The effects of viscous dissipation and radiation in the energy equation and the effect of Hall current in Ohm's law are neglected because they are expected to be small in the case of an arc of low power gradient and because the simplifications thus afforded are essential to the subsequent development of the solutions.

In the present two-dimensional situation created by the assumption of a uniform arc column, $\mathbf{E}$ is assumed to have only one component E_z which is constant by Faraday's law $\nabla \times \mathbf{E} = 0$ and therefore equal to E_∞, the applied electric field. Also it is assumed that $\mathbf{v}$ and $\mathbf{B}$ have no z-components and do not depend on z. Then $\mathbf{j}$ has only the component j_z which is independent of z. It is possible by virtue of (1) and (5) and the two-dimensionality of the problem to introduce a velocity stream function Ψ and a magnetic stream function A; in order that Ψ is the velocity stream function both inside and outside the arc we define it so that

$$\rho v_x = \frac{\partial \Psi}{\partial y}, \quad \rho v_y = -\frac{\partial \Psi}{\partial x}; \quad \rho v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \quad \rho v_\theta = -\frac{\partial \Psi}{\partial r} \tag{1a}$$

(we quote relations in both rectangular co-ordinates x, y and polar co-ordinates r, θ with the origin at the centre of the arc, since we shall use both sets of co-ordinates subsequently); the magnetic stream function is identical with the z-component of the magnetic vector potential $\mathbf{A}$ defined by $\mathbf{B} = \nabla \times \mathbf{A}, \nabla . \mathbf{A} = 0$, and satisfies the relations

$$B_x = \frac{\partial A}{\partial y}, \quad B_y = -\frac{\partial A}{\partial x}; \quad B_r = \frac{1}{r}\frac{\partial A}{\partial \theta}, \quad B_\theta = -\frac{\partial A}{\partial r}. \tag{5a}$$

It is convenient to use (5*a*) and write equations (2), (3), (4), (6) as

$$\rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p+\eta\nabla^2\mathbf{v}+j_z\nabla A, \tag{2a}$$

$$\rho\mathbf{v}.\nabla[(c_p/k)\phi+\tfrac{1}{2}v^2] = E_\infty j_z+\nabla^2\phi, \tag{3a}$$

$$\nabla^2 A = -\mu j_z, \tag{4a}$$

$$j_z = \sigma(E_\infty-\mathbf{v}.\nabla A). \tag{6a}$$

It is not worth while replacing $\mathbf{v}$ by Ψ in these equations at this stage.

The mathematical problem is to solve (2*a*), (3*a*), (4*a*) and (6*a*) outside the arc and inside the arc using the values of the constants as specified below:

$$\left.\begin{array}{lllll} \text{outside the arc } (\phi < \hat{\phi}): & \sigma = 0, & \rho=\rho_2, & \eta=\eta_2, & (c_p/k) = (c_p/k)_2; \\ \text{inside the arc } (\phi > \hat{\phi}): & \sigma = \sigma_1, & \rho=\rho_1, & \eta=\eta_1, & (c_p/k) = (c_p/k)_1; \end{array}\right\} \tag{7}$$

and then to join these two solutions by satisfying suitable boundary conditions at the arc periphery.

2.2. *Boundary conditions*

The imposed boundary conditions are $\phi = \phi_\infty$, $B_x = 0$, $B_y = B_\infty$, $v_x = U$, $v_y = 0$, $p = p_\infty$ at $r = \infty$, and $\nabla\phi = 0$ at $r = 0$ since the origin is taken at the point of maximum temperature. The conditions applied at the periphery (later shown to be a circle) preserve the continuity of heat-flux potential ϕ (and hence temperature T) and heat flux $\nabla\phi$, the continuity of magnetic stream function A and magnetic field $\mathbf{B}$, and the continuity of tangential velocity v_θ, mass flux ρv_r (and hence velocity stream function Ψ), tangential stress $\tau_{r\theta}$ defined by $\tau_{r\theta} = \eta(\partial v_\theta/\partial r - v_\theta/r + \partial v_r/r\,\partial\theta)$ and normal stress $(-p+\tau_{rr})$ where $\tau_{rr} = 2\eta\,\partial v_r/\partial r$. Note that the Maxwell stresses are continuous, since the components of the magnetic field are continuous.

The conditions on the flow at the periphery are physically consistent from a continuum viewpoint and are based on experience with various combinations of conditions which shows that they lead to plausible flow patterns satisfying appropriate integral relations. However, they do not lead to a unique flow pattern for a given set of values of current, imposed velocity and ambient pressure and temperature. One further boundary condition on the flow at the periphery is required to achieve uniqueness, but since it is not clear what form this condition should take it is left open.

2.3. *Approximate equations for an arc of low power gradient*

Outside the arc (region 2) we make the Oseen approximation in which the imposed velocity is used instead of the local velocity in the convective term in the energy equation and in the factor multiplying the velocity gradient in the inertial term of the momentum equation. When the appropriate constants from (7) are used, equations (2*a*), (3*a*), (4*a*), (6*a*) become

$$\rho_2 U(\partial\mathbf{v}/\partial x) = -\nabla p+\eta_2\nabla^2\mathbf{v}, \tag{2b}$$

$$\rho_2 U(c_p/k)_2(\partial\phi/\partial x) = \nabla^2\phi, \tag{3b}$$

$$\nabla^2 A = 0, \tag{4b}$$

$$j_z = 0. \tag{6b}$$

The equations from which $\mathbf{v}$, p, ϕ and A must be derived are now uncoupled and linearized.

Inside the arc (region 1) we make the Stokes approximation, in which the convective term in the energy equation and the inertial term in the momentum equation are neglected, together with the assumption that the induced electric field $\mathbf{v}.\nabla A$ is small compared with the applied field E_∞. When the appropriate constants from (7) are used, equations (2a), (3a), (4a), (6a) reduce to

$$0 = -\nabla p + \eta_1 \nabla^2 \mathbf{v} + j_z \nabla A, \tag{2c}$$

$$0 = E_\infty j_z + \nabla^2 \phi, \tag{3c}$$

$$\nabla^2 A = -\mu j_z, \tag{4c}$$

$$j_z = \sigma_1 E_\infty. \tag{6c}$$

The form (6c) of Ohm's law is largely responsible for the simple solutions derived subsequently. By neglecting $\mathbf{v}.\nabla A$ compared with E_∞ the last obstacle to completely uncoupled equations is removed, and so is a vorticity interaction which is a chief concern of classical magnetohydrodynamics (Shercliff 1965). For, since σ_1 is constant, equations (2c) and (6c) lead to

$$0 = -\nabla(p - \sigma_1 E_\infty A) + \eta_1 \nabla^2 \mathbf{v}, \tag{2d}$$

and hence by taking the curl of this equation and using Ψ from (1a) we have

$$\nabla^4 \Psi = 0. \tag{2e}$$

Also, when (6c) is used equations (3c) and (4c) become simply

$$\nabla^2 \phi = -\sigma_1 E_\infty^2, \tag{3d}$$

$$\nabla^2 A = -\mu \sigma_1 E_\infty. \tag{4d}$$

By these approximations the problems for the temperature and the magnetic field are separated from each other and from the problem involving the velocity and the pressure. We therefore treat them separately in the following sections. In the derivation of the solutions we follow the procedure, which appears to be basic in dealing with arc problems, of treating heat transfer as the primary feature, since it determines the size and shape of the arc, and regarding the situation outside the arc as controlling that inside the arc.

3. Temperature

3.1. *Solutions outside and inside the arc*

Outside the arc ϕ must satisfy equation (3b). A general solution of this equation exists in the form of an infinite series involving modified Bessel functions but since we believe, following the view of Proudman & Pearson (1957), that the nature of Oseen's approximation makes it superfluous to seek high-order solutions we consider only the first-order solution. When the condition $\phi \to \phi_\infty$ as $r \to \infty$ is satisfied this solution may be written as

$$\phi = \phi_\infty + (Q/2\pi) \exp(P r_2 \zeta \cos\theta) K_0(P r_2 \zeta), \tag{8}$$

where K_0 is the modified Bessel function of the second kind of zero order, ζ is a non-dimensional Oseen variable defined by

$$\zeta = \rho_2 U r / 2\eta_2, \tag{9}$$

Pr_2 is the Prandtl number defined by

$$Pr_2 = (c_p/k)_2 \eta_2, \tag{10}$$

and Q is a constant. We assume that Pr_2 is $O(1)$. When ζ is large compared with unity the appropriate asymptotic form of ϕ shows that the uniform temperature at infinity is approached through a laminar thermal wake and identifies Q as the enthalpy flux in the wake

$$Q = \rho_2 U (c_p/k)_2 \int_{-\infty}^{+\infty} (\phi - \phi_\infty)\, dy.$$

We now assume that ζ is small near the arc. When ζ is small compared with unity the solution (8) for ϕ may be written as

$$\phi = \phi_\infty - (Q/2\pi) \log \left(\tfrac{1}{2} e^{\gamma} Pr_2 \zeta\right) + \ldots, \tag{11}$$

where γ is Euler's constant and $e^{\gamma} = 1{\cdot}781$; this has the form of a pure conduction solution in which the temperature depends only on the radial co-ordinate, the heat flux vector therefore has only a radial component, and the total amount of heat conducted across any circle is constant and equal to Q. Since the arc periphery is defined as an isothermal it follows that the arc periphery is a circle. If we denote the radius of the arc periphery by $r = \hat{R}$ and introduce the Stokes variable ξ defined by

$$\xi = r/\hat{R} \tag{12}$$

then, from (11), (9) and (12), near the arc ϕ may be expressed as

$$\phi = \phi_\infty - (Q/2\pi) \log \left(\tfrac{1}{8} e^{\gamma} Pr_2 Re_2 \xi\right) + \ldots, \tag{13}$$

where the Reynolds number Re_2 is defined by

$$Re_2 = \rho_2 U 2\hat{R}/\eta_2. \tag{14}$$

By putting $\phi = \hat{\phi}$ (a constant, assumed to be deducible from experiments) at $\xi = 1$ in equation (13) and rearranging we obtain

$$\frac{Q}{4\pi(\hat{\phi} - \phi_\infty)} = \frac{1}{\log \left(\tfrac{1}{8} e^{\gamma} Pr_2 Re_2\right)^{-2}}. \tag{15}$$

Equation (15) is the basic relation expressing the heat transfer from the arc to the flow. It is the same as that for the heat transfer from a hot solid circular cylinder at low Reynolds numbers, first given by Cole & Roshko (1954), since $Q/4\pi(\hat{\phi} - \phi_\infty) = \tfrac{1}{4} Nu$ where the Nusselt number Nu is defined by

$$Nu = (Q/2\pi\hat{R})(2\hat{R})/(\hat{\phi} - \phi_\infty).$$

If we now introduce the notation

$$N = \frac{Q}{4\pi(\hat{\phi} - \phi_\infty)}, \tag{16}$$

$$S = \left(\tfrac{1}{8} e^{\gamma} Pr_2 Re_2\right)^2, \tag{17}$$

then we have
$$S = \exp(-1/N). \tag{18}$$

This is identical with the external heat transfer relation for a wall-stabilized static arc in a circular tube (Lord 1964) for which $S = (\hat{R}/\tilde{r})^2$ if $\tilde{r}$ is the radius of the tube. It therefore follows by using (17) and (14) that, as far as the temperature and heat transfer near the arc are concerned, the arc in an imposed flow of low Reynolds number is equivalent to a static arc in a circular tube of radius $\tilde{r}$ given by
$$\tilde{r} = \frac{4\eta_2}{e^{\gamma} Pr_2 \rho_2 U}. \tag{19}$$

We regard $\tilde{r}$ given by (19) as the fundamental length scale in the present problem. It replaces the expression suggested by Lord (1964), which however differs from it only by an empirical constant factor.

Inside the arc, ϕ must satisfy equation ($3d$) and since the arc periphery is a circle and the heat flux vector just outside the arc is radial, the appropriate solution for ϕ, which is non-singular at $r = 0$ and satisfies $\phi = \hat{\phi}$ at $\xi = 1$ is
$$\phi = \hat{\phi} + \tfrac{1}{4}E_\infty^2 \sigma_1 \hat{R}^2(1-\xi^2). \tag{20}$$

3.2. *Boundary conditions at the arc periphery*

The stipulation that $\phi = \hat{\phi}$ at the arc periphery in both the external and internal solutions is part of the joining process at the periphery, which is completed by equating the values of $d\phi/dr$ on each side of the periphery; this leads to the result
$$Q = E_\infty^2 \sigma_1 \pi \hat{R}^2. \tag{21}$$

It then follows from (20) and (21) that ϕ_0, the value of ϕ at the centre of the arc, is given by
$$4\pi(\phi_0 - \hat{\phi}) = Q, \tag{22}$$

which may be rewritten as
$$(\phi_0 - \hat{\phi})/(\hat{\phi} - \phi_\infty) = N. \tag{23}$$

Therefore, since N is small compared with one, the difference of heat-flux potential between the centre of the arc and the periphery is small compared with that between the periphery and the ambient flow. Hence, in the present approximation the value of all gas properties within the arc, with the important exception of the electric conductivity as discussed later, may be taken at the peripheral temperature. Therefore, the constancy of ρ, η and (c_p/k) inside the arc now becomes a valid approximation instead of an assumption.

The result (21) may be put in a simpler and more familiar form by noting that, from Ohm's law ($6c$), the total current I is given by
$$I = E_\infty \sigma_1 \pi \hat{R}^2. \tag{24}$$

Hence, from (21) and (24),
$$E_\infty I = Q, \tag{25}$$

which, in the absence of radiation, is a general result independent of the form of approximation used for the electric conductivity.

4. Magnetic field

Outside the arc the magnetic stream function A satisfies equation (4*b*) and the appropriate solution satisfying the boundary conditions $B_x = 0$, $B_y = B_\infty$ at $r = \infty$ is that given by the superposition of the applied field and the induced field due to an infinite cylindrical conductor:

$$A = \text{constant} - (\mu I/2\pi)\log r - B_\infty r\cos\theta. \tag{26}$$

By taking the induced magnetic stream function to have the value $\hat{A}$ at the arc periphery, equation (26) becomes, in terms of the Stokes variable ξ (since the Oseen variable ζ has no significance for the magnetic field),

$$A = \hat{A} - (\mu I/2\pi)\log\xi - B_\infty\hat{R}\xi\cos\theta. \tag{27}$$

Inside the arc, A satisfies equation (4*d*) and, since σ_1 is constant, the appropriate solution is given by the superposition of the applied field and the induced field due to an infinite cylinder of uniform current density:

$$A = \text{constant} - \tfrac{1}{4}\mu E_\infty\sigma_1 r^2 - B_\infty r\cos\theta. \tag{28}$$

By making A and both field components continuous at the periphery it follows from (27) and (28) that inside the arc

$$A = \hat{A} + (\mu I/4\pi)(1-\xi^2) - B_\infty\hat{R}\xi\cos\theta. \tag{29}$$

5. Velocity and pressure

5.1. *Solutions outside and inside the arc*

Outside the arc the velocity and pressure satisfy equation (2*b*), the general solution of which, given by Lamb (1932), is

$$\mathbf{v} = \nabla\Phi + (\eta_2/\rho_2 U)\nabla\chi - \chi\mathbf{i}, \tag{30}$$

$$p = p_\infty - \rho_2 U\{(\partial\Phi/\partial x) - U\}, \tag{31}$$

where $\mathbf{i}$ is a unit vector in the x-direction and Φ and χ satisfy the equations

$$\nabla^2\Phi = 0, \tag{32}$$

$$\nabla^2\chi - \frac{\rho_2 U}{\eta_2}\frac{\partial\chi}{\partial x} = 0. \tag{33}$$

From the general solutions of equations (32) and (33) which exist in series form we need consider only the first-order solutions:

$$\Phi = U[r\cos\theta + \beta\log r + \beta_1\cos\theta/r], \tag{34}$$

$$\chi = (D/2\pi\eta_2)\exp(\zeta\cos\theta)K_0(\zeta), \tag{35}$$

where β, β_1 and D are dimensional constants and ζ is the Oseen variable given by (9). When ζ is large compared with unity the asymptotic form of χ shows that there is a laminar viscous wake behind the arc and identifies D as the drag on the arc:

$$D = \rho_2 U\int_{-\infty}^{+\infty}(U - v_x)\,dy.$$

When ζ is small compared with unity equation (35) may be approximated by

$$\chi = -(D/2\pi\eta_2)(1+\zeta\cos\theta)\log(\tfrac{1}{2}e^{\gamma}\zeta)+\ldots, \tag{36}$$

where, in order to obtain a consistent approximation for the velocity components, it is necessary to include an extra term compared with the corresponding approximate form for ϕ. After some manipulation of (30), (1*a*), (34) and (36) the stream function Ψ may be shown to be given, for small ζ, by

$$\Psi = \rho_2 U\left[\left(\beta - \frac{D}{2\pi\rho_2 U^2}\right)\theta + \left\{1 - \frac{\frac{1}{4}\rho_2^2 U^2\beta_1/\eta_2^2}{\zeta^2} - \frac{D}{8\pi\eta_2 U}[2-\log(\tfrac{1}{2}e^{\gamma}\zeta)^2]\right\} r\sin\theta\right]. \tag{37}$$

By stipulating that there is no net source- or sink-effect in the arc we obtain

$$\beta = D/2\pi\rho_2 U^2, \tag{38}$$

and hence from (37), (38) and (9)

$$\Psi = 2\eta_2\left\{1 - \frac{\frac{1}{4}\rho_2^2 U^2\beta_1/\eta_2^2}{\zeta^2} - \frac{D}{8\pi\eta_2 U}[2-\log(\tfrac{1}{2}e^{\gamma}\zeta)^2]\right\}\zeta\sin\theta. \tag{39}$$

In terms of $\xi = \zeta/\frac{1}{4}Re_2$, Ψ becomes

$$\Psi = \rho_2 U\hat{R}\left\{1 - \frac{\beta_1/\hat{R}^2}{\xi^2} - \frac{D}{8\pi\eta_2 U}[2-\log(\tfrac{1}{8}e^{\gamma}Re_2)^2 - \log\xi^2]\right\}\xi\sin\theta. \tag{40}$$

We now introduce the non-dimensional parameter Δ defined by

$$\Delta = D/8\pi\eta_2 U, \tag{41}$$

noting that, in terms of the drag coefficient $C_{D2} = D/\frac{1}{2}\rho_2 U^2(2\hat{R})$, $\Delta = C_{D2}Re_2/16\pi$; Δ is small compared with unity. If we now write

$$\beta_1 = -\Delta\hat{R}^2 c, \tag{42}$$

$$\Delta = \frac{1}{\log(\frac{1}{8}e^{\gamma}Re_2)^{-2}+2-d}, \tag{43}$$

the result for Ψ becomes

$$\frac{\Psi}{\rho_2 U\hat{R}} = \Delta\left(\log\xi^2 + \frac{c}{\xi^2} - d\right)\xi\sin\theta. \tag{44}$$

The expression for the external flow near the arc, which is recognizable as the appropriate solution of the Stokes equations, therefore involves two arbitrary parameters, here denoted by c and d, Δ being related to d by (43); c and d are not restricted to $O(1)$, and $\Psi/\rho_2 U\hat{R}$ is not restricted to $O(\Delta)$, although this is most often the case. The parameter d is convenient for expressing the stream function in a simple form, but a more convenient parameter from the point of view of the expression of the drag is δ defined by

$$d = 1-\delta, \tag{45}$$

for then the non-dimensional drag parameter Δ is given by

$$\Delta = \frac{1}{\log(\frac{1}{8}e^{\gamma}Re_2)^{-2}+1+\delta} \tag{46}$$

and when $\delta = 0$ this reduces to the expression for the drag of a solid body. It is usual in low Reynolds number theory to write (Lagerstrom 1964)

$$\epsilon = 1/[\tfrac{1}{2} - \log(\tfrac{1}{8}e^{\gamma} Re_2)]$$

and in terms of ϵ we have $\Delta = \frac{1}{2}\epsilon/(1+\frac{1}{2}\epsilon\delta)$; for a solid body $\Delta = \frac{1}{2}\epsilon$.

Inside the arc the stream function satisfies equation (2e) and the required solution is of the form $f(r)\sin\theta$ and is non-singular at $r = 0$. It may be written without loss of generality in terms of ξ as

$$\frac{\Psi}{\rho_2 U\hat{R}} = \Delta\frac{\rho_1}{\rho_2}(a\xi^2 - b)\xi\sin\theta, \tag{47}$$

where a and b are arbitrary constants. Both equations (44) and (47) are special cases of the general Stokes solution of the form $f(r)\sin\theta$.

The solution for the pressure outside the arc follows from (31), (34) and (38) as

$$p = p_\infty - \frac{D}{2\pi}\frac{\cos\theta}{r}, \tag{48}$$

where the term involving β_1 is omitted since it is of higher order in Δ. Inside the arc the pressure is obtained from (2d) and since A is given by (29) and the viscous term gives a constant pressure gradient in the x-direction, the solution is

$$p = p_0 - \tfrac{1}{4}\mu E_\infty^2\sigma_1^2 r^2 - \left[\sigma_1 E_\infty B_\infty + \frac{a}{(\eta_2/\eta_1)}\frac{D}{\pi\hat{R}^2}\right] r\cos\theta, \tag{49}$$

where p_0 is the unknown pressure at the centre of the arc.

5.2. *Boundary conditions at the arc periphery*

We now apply boundary conditions at the arc periphery. Continuity of $v_\theta, \rho v_r$ and $\tau_{r\theta}$ leads to the following expressions for a, b and c in terms of δ:

$$a = \frac{\dfrac{\eta_2}{\eta_1}\left[\left(1+\dfrac{\rho_1}{\rho_2}\right) - \left(1-\dfrac{\rho_1}{\rho_2}\right)\delta\right]}{1+\dfrac{\rho_1}{\rho_2}+2\dfrac{\eta_2\rho_1}{\eta_1\rho_2}}, \tag{50}$$

$$b = \frac{\dfrac{\eta_2}{\eta_1}\left(3+\dfrac{\rho_1}{\rho_2}\right) - \left(2+3\dfrac{\eta_2}{\eta_1}-\dfrac{\eta_2\rho_1}{\eta_1\rho_2}\right)\delta}{1+\dfrac{\rho_1}{\rho_2}+2\dfrac{\eta_2\rho_1}{\eta_1\rho_2}}, \tag{51}$$

$$c = \frac{\left(1+\dfrac{\rho_1}{\rho_2}\right) - \left(1-\dfrac{\rho_1}{\rho_2}\right)\delta}{1+\dfrac{\rho_1}{\rho_2}+2\dfrac{\eta_2\rho_1}{\eta_1\rho_2}}. \tag{52}$$

Continuity of $(-p+\tau_{rr})$ then gives

$$p_0 = p_\infty + \tfrac{1}{4}\mu E_\infty^2\sigma_1^2\hat{R}^2, \tag{53}$$

$$B_\infty I = D; \tag{54}$$

these two integral relations serve to determine p_0 and B_∞. The pressure itself is discontinuous, and so is the vorticity. The relations for a, b and c show that the flow near and within the arc is determined by the viscosity ratio η_2/η_1, the density ratio ρ_1/ρ_2 (both of which depend ultimately on the ambient conditions) and the parameter δ. The effect of the final boundary condition on the flow at the periphery would be to specify δ in terms of η_2/η_1 and ρ_1/ρ_2.

It is illuminating to consider what the foregoing expressions imply with regard to the nature of the arc periphery. This can be done conveniently by defining two non-dimensional parameters which represent the extent to which the flow is slipping around the periphery and the extent to which it is crossing the periphery. We call these parameters the slipperiness and the porousness, denote them by Σ and Π, and define them by

$$\Sigma = \frac{(-v_\theta)_{r=\hat{R},\,\theta=\frac{1}{2}\pi}}{U}, \tag{55}$$

$$\Pi = \frac{(-\rho v_r)_{r=\hat{R},\,\theta=\pi}}{\rho_2 U}. \tag{56}$$

These are special definitions appropriate to the simple symmetrical flows under discussion, and more general definitions suitable for any slippery porous body could be devised. Σ and Π are connected with c and d by $c = 1-\frac{1}{2}(\Sigma-\Pi)/\Delta$ and $d = 1-\frac{1}{2}(\Sigma+\Pi)/\Delta$. Hence, from (43) and the definition of ϵ, Δ may be written as $[1-\frac{1}{2}(\Sigma+\Pi)]\frac{1}{2}\epsilon$, which shows that the drag of a slippery porous body is equal to a factor $[1-\frac{1}{2}(\Sigma+\Pi)]$ times the drag of the corresponding solid body. The drag of a slippery porous body is therefore zero if the slipperiness and porousness satisfy the condition $\Sigma+\Pi = 2$. The continuity of tangential velocity, normal mass flux and tangential stress at the arc periphery leads to the following expressions for Σ and Π in terms of δ:

$$\Sigma = \Delta \frac{2\left(1+\dfrac{\eta_2\rho_1}{\eta_1\rho_2}\right)}{\left(1+\dfrac{\rho_1}{\rho_2}+2\dfrac{\eta_2\rho_1}{\eta_1\rho_2}\right)}\left[\delta+\frac{\dfrac{\eta_2\rho_1}{\eta_1\rho_2}}{\left(1+\dfrac{\eta_2\rho_1}{\eta_1\rho_2}\right)}\right], \tag{57}$$

$$\Pi = \Delta \frac{2\dfrac{\rho_1}{\rho_2}\left(1+\dfrac{\eta_2}{\eta_1}\right)}{\left(1+\dfrac{\rho_1}{\rho_2}+2\dfrac{\eta_2\rho_1}{\eta_1\rho_2}\right)}\left[\delta-\frac{\dfrac{\eta_2}{\eta_1}}{\left(1+\dfrac{\eta_2}{\eta_1}\right)}\right]. \tag{58}$$

Some flow properties of various well-known types of body may be recovered from equations (57) and (58) by specifying, in the first place, relevant material properties of the bodies. A solid body may be regarded as having infinite viscosity, so $\eta_2/\eta_1 = 0$ and hence

$$\Sigma = \Delta[2/(1+\rho_1/\rho_2)]\delta, \quad \Pi = \Delta[2(\rho_1/\rho_2)/(1+\rho_1/\rho_2)]\delta$$

with ρ_1/ρ_2 arbitrary. But a solid body is obviously non-porous, so $\Pi = 0$ and hence $\delta = 0$ and so it follows that $\Sigma = 0$; that is, the no-slip condition applies at the surface of a solid body. Inside a cavity the density is zero, so $\rho_1/\rho_2 = 0$ and hence $\Sigma = \Delta 2\delta$, $\Pi = 0$. Unless a further condition which specifies δ is applied the slipperiness and hence the drag of a cavity remain arbitrary. For a fluid bubble, with arbitrary values of viscosity and density inside and outside the bubble, the condition of zero porousness applies and then $\Pi = 0$ leads to $\delta = (\eta_2/\eta_1)/(1+\eta_2/\eta_1)$ with $\Sigma = \Delta 2(\eta_2/\eta_1)/(1+\eta_2/\eta_1) = \Delta 2\delta$, which latter result is seen to be a general result for non-porous bodies.

Since, in particular, the appropriate boundary conditions for a solid body can be derived by first specifying a material property and then invoking the non-porous condition, the no-slip condition following as a consequence, it seems reasonable to expect that a similar situation exists for an arc and that the boundary conditions for an arc can be obtained by specifying a material property plus a further flow condition. It is this last condition which is unknown and which we leave open.

With regard to material properties, whereas in the above discussion of special bodies η_2/η_1 and ρ_1/ρ_2 are regarded as independent of each other, for an arc they are not independent since they are both functions of the ambient temperature. Also, by their nature as mean values η_2 and ρ_1 cannot be zero and η_1 and ρ_2 cannot be infinite, and so it follows that neither η_2/η_1 nor ρ_1/ρ_2 can be zero. Hence, in the present mathematical model at least, the cases of the solid body and the cavity are excluded from the arc.

5.3. *Examples of possible flow patterns*

In order to illustrate the effect of the final boundary condition on the flow near an arc we assume the gas to be perfect with constant specific heats, constant Prandtl number and thermal conductivity proportional to temperature, and take mean values outside the arc at the ambient pressure and the mean temperature. Then it may be shown that

$$\frac{\eta_2}{\eta_1} = \frac{\rho_1}{\rho_2} = \frac{1}{2}\left(1+\frac{T_\infty}{\hat{T}}\right). \tag{59}$$

We consider eight examples of possible flow patterns, the relevant mathematical details of which are given in table 1 below; for completeness, the values of δ relevant to the various examples are given in their general form in terms of η_2/η_1 and ρ_1/ρ_2 rather than in terms of $T_\infty/\hat{T}$. The flows are illustrated, for $T_\infty/\hat{T} = 0$ and $T_\infty/\hat{T} = 1$, in figure 2. It is significant that all the conditions give $\delta = O(1)$ in general, although $\delta \to \infty$ as $T_\infty/\hat{T} \to 1$ in example 1 and $\delta \to -\infty$ as $T_\infty/\hat{T} \to 1$ in example 8. In general, then, $\Delta = O(\frac{1}{2}\epsilon)$, but in the above cases the drag is zero, Σ and Π being $O(1)$. Otherwise Σ and Π are $O(\Delta)$ throughout. Example 8 is an oddity, however, because the drag becomes infinite when $T_\infty/\hat{T} = 1-2\epsilon \ldots$ so $\delta = -1/(\frac{1}{2}\epsilon)$. Example 2 is the only case in which the final boundary condition is natural rather than contrived in the sense that some flow feature is enforced, although examples 5 and 7 are realistic cases in themselves (being the cases of the fluid bubble and the porous body respectively). Examples 1 and 2 show direct flow through the arc without stagnation points. Examples 3–5 cover the whole range of flows in which the arc has an internal closed streamline. Examples 6–8 all show the main flow diverted away from the arc. It is interesting that when a closed streamline exists it is a circle and therefore an isothermal. The effect of the final boundary condition predominates over the effect of $T_\infty/\hat{T}$, being a maximum when $T_\infty/\hat{T} = 1$. Note that negative slipperiness and negative porousness are possible and imply reversal of flow direction around and across the periphery respectively; these do not occur at the same value of δ.

Ex.	Description	Condition	Value of δ
1	The flow inside the arc periphery is uniform	$a = 0$	$\left(1+\frac{\rho_1}{\rho_2}\right)\Big/\left(1-\frac{\rho_1}{\rho_2}\right)$
2	The pressure is continuous across the arc periphery	$c = \frac{1}{2}$	$\dfrac{1+\frac{\rho_1}{\rho_2}-2\frac{\eta_2\rho_1}{\eta_1\rho_2}}{2\left(1-\frac{\rho_1}{\rho_2}\right)}$
3	The centre of the arc is a stagnation point	$b = 0$	$\dfrac{\frac{\eta_2}{\eta_1}\left(3+\frac{\rho_1}{\rho_2}\right)}{2+3\frac{\eta_2}{\eta_1}-\frac{\eta_2\rho_1}{\eta_1\rho_2}}$
4	The flow has a closed streamline inside the arc periphery	$\delta_4 = \frac{1}{2}(\delta_3+\delta_5)$	$\dfrac{\frac{\eta_2}{\eta_1}\left(5+6\frac{\eta_2}{\eta_1}+\frac{\rho_1}{\rho_2}\right)}{2\left(1+\frac{\eta_2}{\eta_1}\right)\left(2+3\frac{\eta_2}{\eta_1}-\frac{\eta_2\rho_1}{\eta_1\rho_2}\right)}$
5	The periphery is a streamline	$\Pi = 0$	$\frac{\eta_2}{\eta_1}\Big/\left(1+\frac{\eta_2}{\eta_1}\right)$
6	The arc is the slippery porous body equivalent to a solid body	$\delta = 0$	0
7	The streamlines are normal to the arc periphery	$\Sigma = 0$	$\dfrac{-\frac{\eta_2\rho_1}{\eta_1\rho_2}}{1+\frac{\eta_2\rho_1}{\eta_1\rho_2}}$
8	The flow just outside the arc periphery is uniform	$c = 1$	$-2\frac{\eta_2\rho_1}{\eta_1\rho_2}\Big/\left(1-\frac{\rho_1}{\rho_2}\right)$

TABLE 1

The range of flows covered by the present solution includes many which have been conjectured in the past: examples 1, 2, Broadbent (1965*a*); example 3, Goldsworthy (private communication); example 4, Thiene, Chambers & Jaskowsky (1961), Broadbent (private communication); example 5, Roman & Myers (1966), Kuethe, Harvey & Nicolai (1967), Hodnett (1967); examples 6, 7, 8, Schrade (1965 and private communication). This work therefore shows clearly how the mathematical difference between these various conjectures lies in the form of the final boundary condition at the arc periphery.

Two further comments seem worth recording. First, an internal axial flow through a straight arc (which, if a uniform flow, could be accommodated simply by introducing radial and azimuthal components of electric field to cancel the induced radial and azimuthal currents) somewhat in accordance with the ideas of Kuethe *et al.* (1967) would tend to lead to the non-porous periphery of example 5. Secondly, whatever the flow pattern for a straight arc proves to be, the other flow patterns might be appropriate for columns of different curvatures, in which the induced self-magnetic field due to column curvature modifies the applied magnetic field (Schrade 1965).

6. Electric characteristic

In order to derive the electric characteristic it is necessary to introduce a relation between the electric conductivity and the heat-flux potential. We take

$$\sigma_1 = (2/2{\cdot}405)^2 \alpha(\phi_0 - \hat{\phi}), \tag{60}$$

where α is an empirical constant; the number 2·405 represents the first zero of the Bessel function J_0. It may be shown that this choice of σ_1 enables the correct electric characteristic for a linear σ–ϕ relation, which is all that is required for an arc of low power gradient, to be obtained by the present approximate method in which a constant conductivity is assumed. The result (22) for the temperature at the centre of the arc and the result (24) for the total current are not, however, those given by a linear σ–ϕ relation.

We now define the non-dimensional parameters J and K by

$$J = \frac{E_\infty I}{4\pi(\hat{\phi} - \phi_\infty)}, \tag{61}$$

$$K = \frac{I/E_\infty}{[(2/2{\cdot}405)^2 \alpha(\hat{\phi} - \phi_\infty)][\pi(4\eta_2/e^\gamma Pr_2 \rho_2 U)^2]}, \tag{62}$$

and note that when the gas composition is fixed and the ambient temperature is constant then, provided $\hat{\phi}$ (and $\hat{T}$) are regarded as independent of ambient pressure p_∞, J is proportional to $E_\infty I$ and K is proportional to $p_\infty^2 U^2 I/E_\infty$. It follows from (25), (61) and (16) that

$$J = N, \tag{63}$$

and from (24), (60), (62), (23) and (17) it follows that

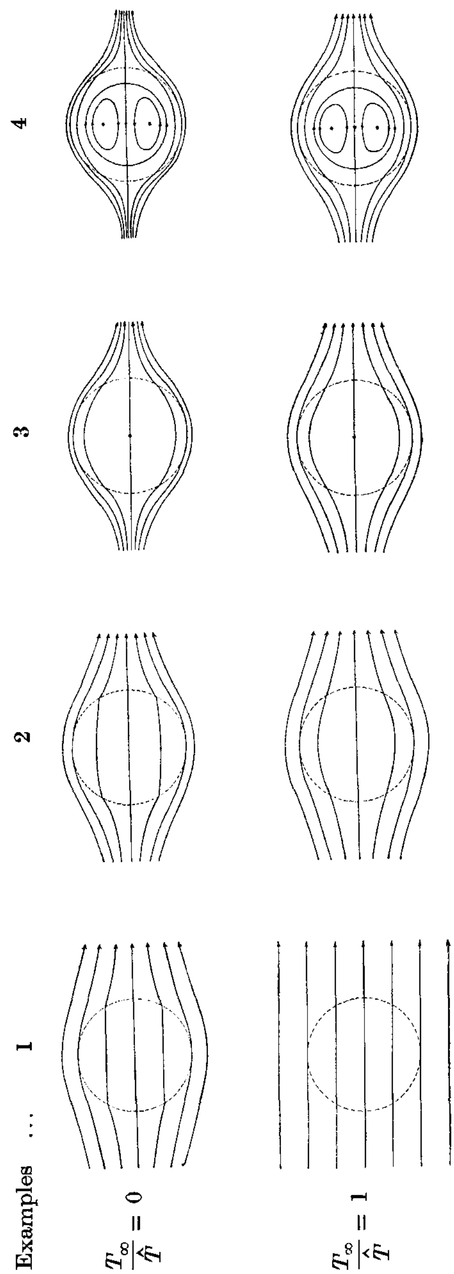
$$K = NS. \tag{64}$$

Hence, by using (18) in combination with (64) and (63) the electric characteristic is given in the form

$$K = J \exp(-1/J). \tag{65}$$

The electric characteristic of a low power arc in a magnetic field is therefore precisely the same as the electric characteristic of a wall-stabilized static arc.

It is more usual in practice to regard the electric characteristic as a relation between the electric field and the current, and this can be obtained in non-dimensional form from (65) by using the non-dimensional parameters F and G defined by $F = (J/K)^{\frac{1}{2}}$, $G = (JK)^{\frac{1}{2}}$. When the gas composition is fixed and the ambient temperature is constant, F is proportional to $E_\infty/p_\infty U$ and G is proportional to $p_\infty UI$.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$\frac{T_\infty}{\hat{T}} = 0$
$\frac{T_\infty}{\hat{T}} = 1$

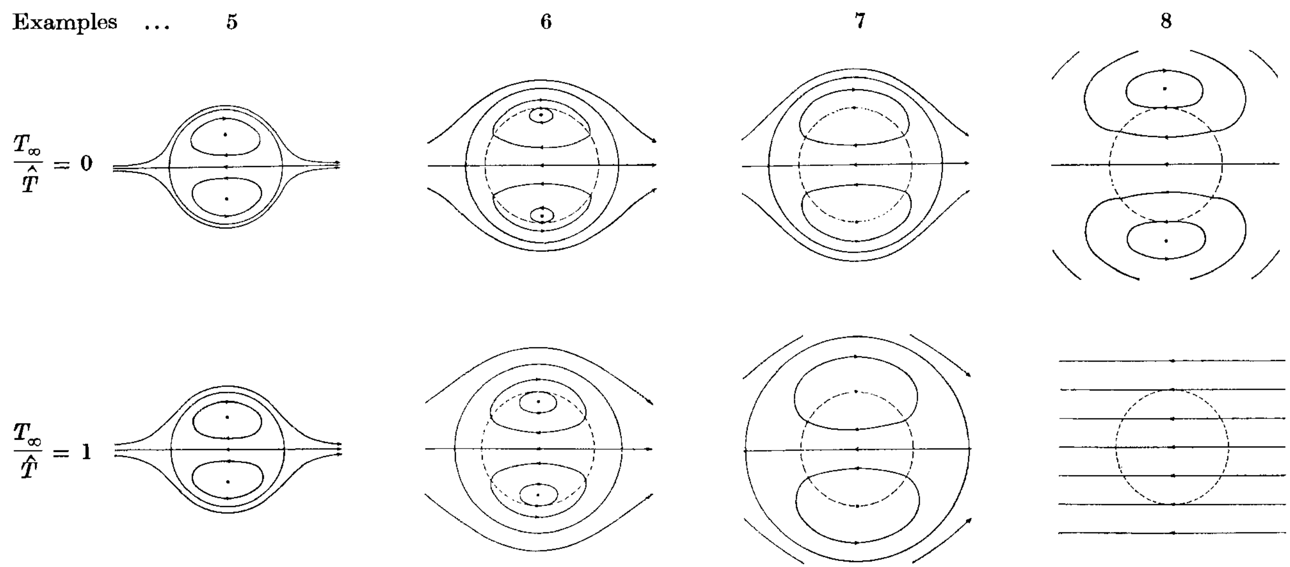

FIGURE 2. Some possible flow patterns in the extreme cases when $T_\infty/\hat{T} = 0$ and $T_\infty/\hat{T} = 1$: (1) uniform flow inside periphery; (2) pressure continuous across periphery; (3) centre is stagnation point; (4) closed streamline inside periphery; (5) periphery is streamline; (6) drag is that of solid body; (7) streamlines normal to periphery; (8) uniform flow just outside periphery.

The non-dimensional parameters J, K and F, G are established here for arcs of low power and may be expected to be significant for arcs of higher power also. The electric characteristic will not then be given by (65), however, as demonstrated by Lord (1967). Parameters of this form were previously given by Lord (1964) and reported by Broadbent (1965*b*). The previously-suggested similarity law that $E_\infty I$ is a function of $p_\infty^2 U^2 I/E_\infty$ (in the absence of radiation) is therefore confirmed in the present work for arcs of low power gradient.

7. Magnetic characteristic

To determine the magnetic characteristic we introduce the non-dimensional parameter L defined by

$$L = B_\infty I/8\pi\eta_2 U \tag{66}$$

and note that when the gas composition is fixed and the ambient temperature is constant L is proportional to $B_\infty I/U$. It then follows from (54), (66) and (41) that

$$L = \Delta, \tag{67}$$

and hence from (46), (17), (18) and (63) the magnetic characteristic is

$$L = \frac{1}{(1/J)+1-\log Pr_2^{-2}+\delta}. \tag{68}$$

The magnetic characteristic is therefore not determined to the extent that δ is unknown, and it follows that the magnetic characteristic is dependent on the final boundary condition at the arc periphery. Since J is a function of K from (65), it follows that L is a function of K, and depends on Pr_2 and δ also.

It may be noted that if $\delta = O(1)$ then equation (68) may be expanded as a series in powers of J, giving $L = J-(1-\log Pr_2^{-2}+\delta)J^2+\dots$, and from (61) and (66) it follows that the first-order result $L = J$ may be written as

$$B_\infty/E_\infty = [2\eta_2/(\hat{\phi}-\phi_\infty)]\,U;$$

this is of the same form as a result given by Otis (1967) with a different coefficient of proportionality between B_∞/E_∞ and U.

The magnetic characteristic may be expressed as a relation between the magnetic field and the current by defining the non-dimensional parameter H by $H = L/G$ and then by making the appropriate substitutions in (68) to obtain the relation between H and G. When the gas composition is fixed and the ambient temperature is constant, H is proportional to $B_\infty/p_\infty U^2$.

A parameter of the form of L was previously given by Lord & Broadbent (1965) and reported by Broadbent (1965*b*), and a similarity law, $B_\infty I/U$ is a function of $p_\infty^2 U^2 I/E_\infty$, suggested. The present work confirms this law for arcs of low power gradient, provided the ambient temperature is kept fixed so that δ is unchanged, in spite of its showing simultaneously that the conception of an arc as a solid conductor, on which the previous law was based, is an illusion.

8. Connexion with experiment

An analysis (Lord 1967) of the experimental electric characteristic of a horizontal free-burning arc in air and nitrogen at one atmosphere pressure, given by King (1961), indicates that there exists a range of power gradient (roughly from 4×10^3 W/m to 6×10^3 W/m in the case of the free-burning arc) in which the discharge is in thermal equilibrium and the method of low power gradient applies. Moreover, the analysis for the free-burning arc when used in conjunction with a similar analysis for the experimental characteristic of the wall-stabilized arc in nitrogen at one atmosphere, given by Maecker (1959), enables certain mean properties of the gas outside the arc to be deduced (Lord 1967). It is suggested that a similar situation may exist for the arc in a transverse magnetic field, and that experiments of the kind performed by Roman & Myers (1966), but at lower levels of current, velocity and magnetic field (say, currents about 1 amp, velocities of order 1 m/sec and fields of order 10^{-3} Wb/m^2), might provide significant evidence on arc structure and also useful information on effective gas properties. (Note the distinction between the previous choice of mean gas properties in order to provide illustrative examples and the present suggestion for a method of deducing mean gas properties from suitable experimental measurements.) It is interesting that the experiments of Roman & Myers (1966) already indicate that an arc of low power is of circular cross-section and that the heat transfer from such an arc is similar to that from a solid body of the same dimensions, which are both features of the present theory.

The proposed method of analysis of experimental characteristics is as follows. The electric characteristic, equation (65), may be written as

$$\log\left(\frac{E_\infty}{U}\right)^2 = \log\left[\frac{\tilde{Q}}{\tilde{\sigma}\pi(4\eta_2/e^\gamma Pr_2\rho_2)^2}\right] + \tilde{Q}\left(\frac{1}{E_\infty I}\right), \tag{69}$$

where $\tilde{Q} = 4\pi(\hat{\phi} - \phi_\infty)$ and $\tilde{\sigma} = (2/2{\cdot}405)^2\alpha(\hat{\phi} - \phi_\infty)$; hence a graph of $\log(E_\infty/U)^2$ against $(1/E_\infty I)$ may be expected to yield first $\tilde{Q}$ and then $\tilde{\sigma}\pi(4\eta_2/e^\gamma Pr_2\rho_2)^2$. The magnetic characteristic, equation (68), may be written as

$$\frac{U}{B_\infty I} = \frac{1 - \log Pr_2^{-2} + \delta}{8\pi\eta_2} + \frac{\tilde{Q}}{8\pi\eta_2}\left(\frac{1}{E_\infty I}\right); \tag{70}$$

hence a graph of $U/B_\infty I$ against $(1/E_\infty I)$ may be expected to yield first η_2 since $\tilde{Q}$ is known, and then $1 - \log Pr_2^{-2} + \delta$. The value of $\tilde{Q}$ should check with the value deduced from the wall-stabilized and free-burning arcs. The other values may be combined with the value of $\tilde{\sigma}$ obtained from the wall-stabilized arc and with a relation connecting η_2/ρ_2 and Pr_2 obtained from the free-burning arc to give ρ_2, Pr_2 and δ. Unless η_1 and ρ_1 were deducible also the value of δ would not itself resolve the difficulty of the flow pattern near the arc, for which flow visualization techniques would be required. However, it should provide some indication, since the examples cited show that there is a general correspondence between the value of δ and the flow pattern whatever the values of η_2/η_1 and ρ_1/ρ_2; from example 2 to example 7, for instance, δ varies between 1 and $-\frac{1}{5}$ when $\eta_2/\eta_1 = \rho_1/\rho_2 = \frac{1}{2}$ and between $\frac{3}{2}$ and $-\frac{1}{2}$ when $\eta_2/\eta_1 = \rho_1/\rho_2 = 1$.

9. Conclusion

A theory, which yields detailed distributions of temperature, magnetic field, velocity and pressure, both inside and outside the arc, is given for an arc in a transverse magnetic field. It applies when the arc has low power gradient. Although the theory involves several artificial features, it is well suited for use in conjunction with experimental measurements.

As far as the overall heat transfer and the electric characteristic of the arc are concerned, the arc in a transverse magnetic field is equivalent to a wall-stabilized arc in a circular tube of appropriate radius. The radius of the equivalent tube is regarded as the natural length scale for the arc in a transverse field; it decreases with increase of ambient pressure and flow velocity.

The problem of the flow pattern near the arc is highlighted but not completely resolved, a major question about the flow near the arc periphery being left open. It is argued that this attitude is reasonable in the absence of detailed experimental results on arcs of low power gradient. Although there are some points of resemblance between an arc and a solid body, it seems better to regard an arc as a slippery porous body. The drag and the magnetic characteristic of the arc, and its slipperiness and porousness, depend on the open flow condition, the resolution of which is the most needed improvement of the present theory.

Developments of the theory may be attempted by discarding the assumption of constant gas properties, preferably one at a time in the order of electric conductivity (Hodnett 1967), density outside the arc and viscosity outside the arc and seeking the modified first-order solution. Also it may be extended by retaining the use of constant gas properties and seeking the next higher-order solution of the full equations of motion analytically in the manner of Kaplun (1957) and Proudman & Pearson (1957), or numerically following Dennis & Shimshoni (1964) and Dennis & Smith (1964).

However, there is a limit to the extent to which the methods of low Reynolds number theory may be exploited and progress towards a satisfactory theory of arcs of higher powers will probably be slow. In the meantime the non-dimensional parameters involved in the low power theory, which may be expected to be significant for arcs of higher power also, provide a useful means of analyzing experimental results for high power arcs. Being based on the theory of a uniform column they apply only to long arcs, but they may be combined with the parameters of Yas'ko (1964) and Dautov & Zhukov (1965), which are particularly suitable for short arcs (see Adams, Guile, Lord & Naylor (1967) also), to make possible eventually the presentation in similarity form of the experimental results for arcs of any length.

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